UNIVERSITI PUTRA MALAYSIA

VISUALISING COUPLED ISOTROPIC HARMONIC OSCILLATOR IN NONCOMMUTATIVE PHASE SPACE

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By

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Thesis Submitted to the Department of Physics, Universiti Putra Malaysia, in partial Fulfilment of the Requirements for the Degree of Science Physics with Honours

(February 2022)

DEDICATIONS

To my beloved

Mom, Dad, Brother and Sister

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of (insert the name of the degree)

COUPLED HARMONIC OSCILLATOR IN NONCOMMUTATIVE PHASE SPACE

By

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Harmonic oscillator has been used widely as it provides solvable models in many branches of physics. Furthermore, it often gives a clear illustration of an abstract idea of the behaviour of the system where both oscillators can interact with each other and energy can be transferred between them has been taken into account using a coupled harmonic oscillator system. A point in a phase space can characterise the state of a system in classical mechanics, but Heisenberg's uncertainty principle prevents this in quantum mechanics. In order to solve the new quantum geometry which is known as non commutative phase space, the basic mathematics of noncommutative geometry is introduced in this work. The coupled harmonic oscillators in NC plane were subsequently verified using a general wavefunction for the coupled harmonic oscillator in noncommutative phase space. We can exactly know the eigenvalue and eigenstate of the system if we know the exact general wavefunction of the coupled harmonic oscillator system. Further aiding the understanding, we use Mathematica programming language to simulate the general wavefunction of coupled harmonic oscillator in noncommutative phase space.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah . . . (nama ijazah)

PENGAYUN HARMONIK BERGADUNG DALAM RUANG FASA BUKAN KOMUTATIF

Oleh

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Pengayun harmonik telah digunakan secara meluas kerana ia menyediakan model yang boleh diselesaikan dalam banyak cabang fizik. Tambahan pula, ia sering memberikan gambaran yag jelas tentang idea abstrak dalam penyelesaian masalah. Kelakuan sistem dimana kedua-duanya berinteraksi antara satu sama lain dan tenaga boleh dipindahkan antara mereka telah diambil kira menggunakan pengayun harmonik berganding . Titik dalam ruang fasa boleh mencirikan keadaaan sistem dalam mekanik klasik, tetapi prinsip ketidakpastian Heisenberg menghalangnya dalam konteks mekanik kuantum. Untuk menyelesaikan geometri tidak bermakna yang juga dikenali sebagai ruang fasa bukan komutatif, matematik asas geometri bukan komutatif telah diperkenalkan dalam kerja ini. Pengayun Harmonik berganding dalam satah NC kemudiannya disahkan menggunakan fungsi gelombang am untuk pengayun harmonik berganding dalam ruang asa bukan komutatif. Kita boleh mengetahui dengan tepat nilai eigen dan keadaan eigen sistem sekiranya dapat mengetahui fungsi gelombang am yang tepat bagi pengayun harmonik yang bergandingan. Selanjutnya, kami menggunakan bahasa pengaturcaraan mathematica untuk mensimulasikan fungsi gelombang am pengayun harmonik berganding dalam ruang fasa bukan komutatif.

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APPROVAL

This thesis entitled A COUPLED HARMONIC OSCILLATOR IN NONCOMMUTATIVE PHASE SPACE by NURFARHANA BINTI MOHD NOOR (Matric No.: 198485), was submitted to the Department of Physics, Faculty of Science, Universiti Putra Malaysia and has been accepted as partial fulfilment of the requirement for the degree of Bachelor of Science (Hons.) Major in Physics.

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LIST OF ABBREVIATIONS

TISE	Time Independent Schrödinger Equation
SHO	Simple HarmonicOscillator
CHO	Coupled Harmonic Oscillator
NCQM	Noncommutative Quantum Mechanic
NC	Noncommutative
HOS	Harmonic Oscillator
SVM	Separation Variable Method
PDE	Partial Differential Equation

CHAPTER 1
INTRODUCTION

1.1 Research Background

Quantum mechanics was classified as a new study in physics as it was discovered in the 1920s. In contrast to classical mechanics that describe macroscopic systems, quantum mechanics allows us to describe the submicroscopic particles and unite the concept of energy quantization, wave-particle duality and uncertainty principle that was introduced by Werner Heisenberg and other scientist. Heisenberg uncertainty principle implies that the lower the uncertainty in an object's velocity, the higher the uncertainty in its position, and vice versa.

Other than that, Schrödinger equations have been used to describe the wave function which explains the behaviour of submicroscopic particles. Operator used in this Schrödinger equation is the system's energy known as Hamiltonian. In classical mechanics, the Hamiltonian indicated the sum of the kinetic energy and potential energy. For quantum mechanics, the expression of both energies are then modified into quantum mechanical operators. Harmonic oscillators are very important models that will be used in this work which provide a framework in mathematical treatment for various events such as the vibrational of crystals, the elasticity of objects and also the matter of optical properties.

Schrödinger equation takes into consideration the wave nature of particles such as electrons inside an atom. Three considerations are classical plane wave equations, Broglie's hypothesis of matter waves and energy conservation. There are two types of Schrödinger equation which is Time-Dependent Schrödinger Equation (TISE) which describes the wave function of microscopic particles that will evolve with time and Time-Independent Schrödinger Equation where the time dependence has been removed from the equation. In this study, we will be dealing with stationary states in the Time-Independent Schrödinger Equation as we are still in the context of nonrelativistic quantum mechanics.

Furthermore, the Hamiltonian operator that will be considered is governed by the isotropic and anisotropic harmonic oscillator potential. Harmonic oscillator defines a particle or system that encounters harmonic motion about equilibrium position. For example, an object with mass vibrating on a string. This is also known as Simple Harmonic Oscillator (SHO). The harmonic oscillator also is a model that has been used to solve the physics in different areas due to its mathematical simplicity. Coupled harmonic oscillators (CHO) are a part of this study in which two particles system involve where it is subjected under the influence of isotropic harmonic oscillator potential. We used this coupled harmonic oscillator system in noncommutative phase space to visualise the particle in the three dimensional plotting.

Apart from the above, Quantum mechanics on noncommutative space also known as noncommutative quantum mechanics (NCQM) is also one of the important concepts in this study. It is the one that expresses the position operators do not commute. In this situation, one can consider two operators that involve coordinates and momenta which we know both operators cannot commute. Werner Heisenberg suggested a theory called Heisenberg uncertainty principle where it explains that positions and momenta cannot be measured at the same time. The commutative algebra of classical observables is transformed to the non-commutative (NC) algebra of quantum mechanical observables in the mathematical transition from classical physics to quantum mechanics. Self-adjoint operators working on a Hilbert space (which is usually believed to be some important function space) represent quantum observables. For this study, we used isotropic harmonic oscillators that have the same values of frequency in all three directions (*x*-, *y*- and *z*-axis) in NCQM.

1.2 Problem Statement

In quantum mechanics, Harmonic Oscillator can often be used to study the behaviour of particle potential energy as a function of position in both classical and quantum versions. Coupled harmonic oscillator has been taken into account in this study to know the behaviour of the system where both interact with each other and energy can be transferred between them. Although a lot of study of the coupled harmonic oscillator has been done, but the behaviour of the coupled harmonic oscillator in noncommutative phase space has not been study yet. So, Noncommutative geometry was introduced in this research to solve the puzzle on what is happening to the particle of the coupled harmonic oscillator.

Other than that, there are also less literature ever discussed on the visualisation of the coupled harmonic oscillator in NC plane. General wavefunction for the coupled harmonic oscillator in noncommutative phase space will be derived to verify the coupled harmonic oscillators in NC plane. Thus, we then have to use the general wavefunction of the coupled harmonic oscillator to visualise it in three dimensional plots (3-D plotting).

1.3 Objective Research

- To verify coupled harmonic oscillators in NC plane.
- To visualize the energy eigenvalue and energy eigenstates of coupled harmonic oscillators.

1.4 Organization of Thesis

The thesis consists of five chapters, the Bibliography and The Appendix. We start the thesis with chapter 1 which is a research background on the Schrödinger equations and the relation of harmonic oscillator in noncommutative phase space in Quantum Mechanics. Further this study with Chapter 2 as a literature review where we take a look of other papers or articles about this Couple Isotopic Harmonic Oscillator to boost understanding on these works. Next, in Chapter 3 we are working on the theories and mathematical tools that are going to used for the findings on this thesis. Moreover, in Chapter 4 we verify on how to solve the coupled harmonic oscillator system in Noncommutative phase space and determine its energy levels using some mathematical tools that we have discussed in Chapter 3. Other than that, diagrams and simulation are provided in the Chapter 5 to make the understanding better for the readers about this topic. Last but not least is Chapter 6 where we concluded of what we have learned and found in this study. Appendices are included with some programming on Mathematica that we have used to do some simulations and diagrams.

CHAPTER 2
LITERATURE REVIEW

2.1 History of Schrödinger equation

A French physicist named Louis de Broglie found the features of wave-particle duality in electrons and all particles, not just photons. Electromagnetic radiation has already been proven to have wave-like properties, and other data implies that it also includes particle motion properties. Electric charge transmission, especially for electrons, has been found to have particle motion qualities and is considered to have wave-like characteristics. The particles must be regarded as quanta in relation to the waves in this case. De Broglie then developed and tested his matter-wave hypothesis using 'geometrical' or 'pictorial' methods. He gave no method for obtaining a wave equation for any of the physical systems he showed, making it impossible for him to obtain acceptable equation responses for his waves. (Kozlowski, 2019).

Despite all of these grounds for Schrödinger's interest in de Broglie's work, simply reading it did not inspire him to investigate the issue further. As a result, Debye suggested that Schrödinger deliver a colloquium on the subject. A 'legitimate' theory from de Broglie, according to Schrödinger, would begin with the formulation of such an equation. The subsequent path of history was clearly determined by Schrödinger's peculiar working style. He was always occupied with physics, and as previously stated, he had some enduring thought and interest channels. On the other hand, he never worked to a defined schedule for a long period of time. He'd switch to the most appealing challenge next on offer once he'd completed everything that sounded fascinating and achievable on one topic. Schrödinger was clearly due for a change by the end of 1925, and he gladly accepted the task of finding a wave equation. In Schrödinger's famous series of publications from 1926, the result was the invention of wave mechanics (Kozlowski, 2019).

Due to the invention of the wave mechanics, Erwin Schrödinger is then considered as a pivotal figure in the development of Quantum Mechanics and was known as the "Father of Quantum Mechanics". He contributed a lot in quantum theory as he came out with the wave equation that known as Schrödinger wave equation. In 1933, he was then awarded with the Nobel Prize in Physics. This wave equation describes how the particle's behaviour and how to study the wave function, $\Psi(x)$ of a system that changes over time in Schrödinger equation and it is called as Time-Dependent Schrödinger equation (Davey, 2020). Time-Independent Schrödinger equation produce solutions that are known as stationary states and it can be obtained by separation of variables where time is considered as constant (Belkacemi et. al, 2000).

2.2 Harmonic Oscillator Potential

Physical realization for Harmonic Oscillator can be done via an object mass, m that are bound to a spring that have a force constant,k. The motion of the object sis subjected to the Hooke's Law. Hooke's Law by Robert Hooke stated applied force that are imposed on a string when it is compressed or extended by some distance is proportional to that distance (Dan, 2006) . It is given by this equation :

$$F = ma = m\frac{d^2x}{d\hat{F}} = -kx,$$

where $x = \sin(\omega t)$ is the displacement and $\omega = 2\pi v = \sqrt{k/m}$ is the angular frequency. The potential energy in classical mechanics for harmonic motion is expressed by:

$$V(x) = -\int F dx \; ;$$

$$= -\int(-kx)\, dx;$$

$$= \frac{1}{2}kx^2. \tag{1}$$

The potential that has been discussed in equation (1) cannot be used directly as the classical information is not general enough where we cannot used it to describe the diatomic molecules vibration. So, quantum effect has to be taken into account by taking a step forward using quantum formulation with classical expression of $k = \omega^2 m$

$$V(\mathrm{x}) = \frac{1}{2} m\omega^2 x^2. \quad (2)$$

According to the Jellal et al. (2005), the potential of two coupled harmonic oscillator is then becomes

$$V(\mathrm{x}) = \frac{1}{2}(c_1 x_1^2 + c_2 x_2^2 + c_3 x_1 x_2)$$

where c_1, c_2, and c_3 is said to be constant parameters and they can determine the corresponding density matrix and the Wigner functions.

2.3 The Hamiltonian

In classical mechanics, three standard mechanics which are Newtonian Mechanics, Lagrangian Mechanics and also Hamiltonian Mechanics were introduced. Hamiltonian is the sum of two energy that we called kinetic energy, $T = \frac{1}{2} mv^2$ and potential energy, $U = mgh$. This Hamiltonian is considered important as it can be used to solve problems in quantum mechanics. Hamiltonians are basically used as an operator (Jeffrey, 1966). It can be expressed as:

$$H = p\dot{y} - L, \quad (3)$$

where p is momentum, velocity, $\dot{y} = \frac{\mathrm{dy}}{\mathrm{dt}}$ and we need to find the Lagrangian, L first. Lagrangian can be done by this formula:

$$\begin{aligned} T &= \frac{1}{2} m\dot{y}^2, \\ U &= mgy, \\ L = T - U &= \frac{1}{2} m\dot{y}^2 - mgy \,, \end{aligned} \quad (4)$$

By the equation of Hamiltonian, we will get two other equation which is:

$$\dot{y} = \frac{\partial H}{\partial p};$$
$$\dot{p} = -\frac{\partial H}{\partial y}.$$

Insert equation (2) into equation (1), we will then get:

$$H = m\dot{y}\dot{y} - \frac{1}{2}m\dot{y}^2 + mgy = \frac{1}{2}m\dot{y}^2 + mgy,$$

where $p = m\dot{y}$ and we get the Hamiltonian:

$$H = \frac{p^2}{2m} + mgy. \quad (5)$$

Both Lagrangian and Hamiltonian are important in finding both of the energy which are kinetic energy and potential energy in a particle. This Hamiltonian is considered important as it can be used to solve the problem in quantum mechanics (Jeffrey, 1966). This Hamiltonian in equation (5) is basically used for one-dimensional simple harmonic oscillator.

2.3.1 Two-Dimensional of Isotropic Harmonic Oscillator

Militaru & Munteanu (2013) explained that the dynamics of the two-dimensional of isotropic harmonic oscillator is given by second order ordinary-differential equation

$$\ddot{x} + \omega^2 x = 0,$$

$$\ddot{y} + \omega^2 y = 0,$$

where $\ddot{x} = \frac{\mathrm{d}^2\mathrm{x}}{\mathrm{dt}^2}$, $\ddot{y} = \frac{\mathrm{d}^2\mathrm{q}^2}{\mathrm{dt}^2}$ and $\omega^2 = \frac{k}{m}$. It is often used as a toy model to find conservation laws where we applied the conservation of energy . Suppose that the standard mass of oscillator, m is equals to 1 and k is the elasticity constant. The Lagrangian, $L(x, \dot{x}) = T - V$. Lagrangian for two-dimensional isotropic harmonic oscillator are then expressed as

$$L = \frac{1}{2}[(\dot{x})^2 + (\dot{y})^2] - \frac{k}{2m}[(x)^2 + (y)^2]\,. \qquad (6)$$

Equation (4) then have to be multiplied by the mass, *m* itself. The equation becomes

$$L = \frac{1}{2}m[(\dot{x})^2 + (\dot{y})^2] - \frac{k}{2}[(x)^2 + (y)^2].$$

We will then express the Hamiltonian for two-dimensional of isotropic harmonic oscillator using equation (1). The Hamiltonian becomes

$$H = p(\dot{x} + \dot{y}) - \frac{1}{2}m[(\dot{x})^2 + (\dot{y})^2] - \frac{k}{2}[(x)^2 + (y)^2],$$

$$H = p\dot{x} + p\dot{y} - \frac{1}{2}m[(\dot{x})^2 + (\dot{y})^2] - \frac{k}{2}[(x)^2 + (y)^2],$$

where momentum, $p = mv$.

$$H = m\dot{x}^2 + m\dot{y}^2 - \frac{1}{2}m[(\dot{x})^2 + (\dot{y})^2] - \frac{k}{2}[(x)^2 + (y)^2],$$

$$H = \frac{1}{2}m[(\dot{x})^2 + (\dot{y})^2] - \frac{k}{2}[(x)^2 + (y)^2],$$

or can be expressed in terms of momentum

$$H = \frac{1}{2m}\left(P_{\mathrm{x}}{}^2 + P_{\mathrm{y}}{}^2\right) - \frac{k}{2}[(x)^2 + (y)^2].$$

In quantum mechanics, the Hamiltonian then will be expressed as follows

$$H = \frac{1}{2m}\left(P_{\mathrm{x}}{}^2 + P_{\mathrm{y}}{}^2\right) - \frac{m\omega^2}{2}[(x)^2 + (y)^2]. \qquad (6)$$

For the equation (6), it represents the Hamiltonian equation for two dimensional of isotropic harmonic oscillator. This is important as we want to describe how is the particle behaviour in two directions which are in the *x*-axis direction and in the *y*-axis direction.

2.4 The Center of Mass

The center of mass is very important in this study as it is a very useful reference point especially for calculations in mechanics. In this study we basically use this reference point which is the center of mass and and another relative masses in order to find the coupled harmonic oscillator eigenstate and eigenvalue. In classical mechanics and quantum mechanics, the motion of the centre of mass for a system of particles subjected to external forces is involved in this study. First of all, rigid body of a system is defined in which the relative locations of mass do not change during motion as the interaction forces are very strong. The center of mass is defined as a mass of object or system in a relative position. The equation of position vector for the center of mass, $\boldsymbol{R}_{CM}$ is given as follows

$$M\boldsymbol{R}_{CM} = \sum_{i=1}^{N} m_i \mathbf{r}_i \qquad (7)$$

where r_i is the position vector and m_i is the mass of the *i*-th particle. The total mass of a system as expressed in equation (6) is $M = \sum_{i=1}^{N} m_i$. Equation (6) are then differentiated with respect to time which bring us to

$$M\dot{\boldsymbol{R}}_{CM} = \sum_{i} m_i \dot{\mathbf{r}}_i = \sum_{i} \mathbf{p}_i \equiv \boldsymbol{P} \quad , \qquad (8)$$

where $\boldsymbol{P}$ is total momentum of a system. The center of mass is then are said to be the total momentum, $\boldsymbol{P}$ of a system of particles is the same as that of a particle with mass, M moving with the velocity of the center of mass (Knudsen & Hjorth, 1967). The equation (7) can be simplified in this form

$$\boldsymbol{P} = M\mathbf{v}_{\mathrm{CM}} .$$

2.5 Noncommutative Phase Space

String theory has gained a lot of popularity in the realm of physics. String theory was formerly regarded to be an extraordinary theory since it permitted a wide range of fundamental particles to be represented as different modes of string vibration. Weak forces, electromagnetic forces, strong forces, and gravitational forces are the four forces we must be aware of in physics. String theory has been proposed as a means of combining all of the fundamental forces of the universe. In quantum mechanics, the fundamental particle carries all physical forces, and strong nuclear forces hold ordinary matter together. Gravity a classical geometry theory, hence distance precision is crucial. Gravity, on the other hand, cannot be directly exploited in quantum mechanics.. The physicist then invented new particles of gravity that are called gravitons and tried to incorporate into the quantum field theory but unfortunately such theory does not seem to satisfy needed good calculational properties of quantum field (Gross, 1988).

Noncommutative geometry is one of important concepts in string theory. For certain limit, the string theory has been reduced to a gauge theory on noncommutative space. This brings forward to an intensive research of quantum mechanics on a noncommutative space (Kijanka & Konskinki, 2004). Noncommutative Quantum Mechanics (NCQM) appeal to be very interesting among physicists. Werner Heisenberg proposed that there is a way to remove the infinite quantities in the field theory which is by broadening the noncommutativity of coordinates. The concept of space-time where it proved that some variables commute have been declined by Heisenberg as it become noncommuting and create a different Lie algebra (Gouba, 2016).

Other than that, Kokado, Okamura & Saito (2003) stated that according to their finding, noncommutative factors or parameters make a contribution on the Hall conductivity. There are three cases of noncommutativity for which we can express the Hall conductivity where the coordinate or the momenta are said to be noncommuting or both of the coordinate and momenta would be noncommuting. Harms & Micu (2007) stated that advantages of their method is that it allows them to examine noncommutative momenta while still imposing constraints on the size of the prarameter that describes them. They know that it must be less than $10^{-19}\mathrm{m}^2$ based on the experiments that measure the Hall conductivity. Even in the absence of external magnetic fields, noncommutativity of coordinates or momenta would cause a phase shift as it can be measured with reasonable precision.

CHAPTER 3
THEORY AND METHODOLOGY

3.1 Two-dimensional Isotropic Harmonic Potential by The Harmonic Interaction

Based on equation (3), we can see that the Hamiltonian is related with the kinetic energy and the potential energy, *V(x)* were expressed as shown in equation (1). The Hamiltonian is formulated as follows

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2,$$

where $k = \omega^2 m$, and the Hamiltonian then becomes

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

We will then consider particle of mass, m oscillate under the angular frequency, ω and by the influence of two-dimensional isotropic harmonic oscillator potential where the momentum, p and coordinates are consider in two axis ($\hat{x}_1$-axis and $\hat{y}_1$-axis).

$$\hat{H} = \frac{1}{2m}\left(\hat{P}_x^2 + \hat{P}_y^2\right) + \frac{1}{2}m\omega^2(\hat{x}_1^2 + \hat{y}_1^2). \tag{3.0}$$

Harmonic interaction $\frac{1}{2}k(\hat{x}_1 - \hat{x}_2)^2 + \frac{1}{2}k\left(\hat{y}_1 - \hat{y}_2\right)^2$ (where this subscript 1 and 2 signifies two different particle) were taken placed in the two-dimensional Isotropic Harmonic Oscillator (HOS). According to the equation (3.0), The Hamiltonian for this coupled Harmonic Oscillator becomes

$$\hat{H} = \frac{\hat{p}_{1_x}^2}{2m_1} + \frac{\hat{p}_{1_y}^2}{2m_1} + \frac{1}{2}m_1\omega^2(\hat{x}_1^2 + \hat{y}_1^2) + \frac{\hat{p}_{2_x}^2}{2m_2} + \frac{\hat{p}_{2_y}^2}{2m_2} + \frac{1}{2}m_2\omega^2(\hat{x}_2^2 + \hat{y}_2^2) + \frac{1}{2}k(\hat{x}_1 - \hat{x}_2)^2 + \frac{1}{2}k\left(\hat{y}_1 - \hat{y}_2\right)^2 \tag{3.1}$$

where

$$\hat{P}_x = \hat{p}_{1_x} + \hat{p}_{2x}\,,$$

$$\hat{P}_y = \hat{p}_{1_y} + \hat{p}_{2_y}.$$

Equation (3.1) cannot be solved on its own present form. External energy modes have to separated from all internal modes by transforming it into a center-of-mass coordinate system. By doing this, internal motion is then seen in terms of one particle relative motion with respect to other particles. First, let $\hat{X}$and $\hat{Y}$ referred to the case with absolute motion of center of mass

$$\hat{X} = \frac{m_1\hat{x}_1 + m_2\,\hat{x}_2}{M},$$

$$\hat{Y} = \frac{m_1\hat{y}_1 + m_2\,\hat{y}_2}{M},$$

and $\hat{x}$, $\hat{y}$, $\hat{p}_x$ and $\hat{p}_y$ are referred to two component quantities of relative motion

$$\hat{x} = \hat{x}_1 - \hat{x}_2,$$

$$\hat{y} = \hat{y}_1 - \hat{y}_2,$$

$$\hat{p}_{\mathrm{x}} = \frac{m_2\hat{p}_{1_x} - m_1\hat{p}_{2_x}}{M},$$

$$\hat{p}_{\mathrm{y}} = \frac{m_2\hat{p}_{1_y} - m_1\hat{p}_{2_y}}{M}.$$

where $M = m_1 + m_2$ and reduced mass, $\mu = \frac{m_1 m_2}{m_1+m_2}$. All of the relations leads to

$$\hat{x}_1 = \frac{m_2}{M}\hat{x} + \hat{X}, \qquad \hat{p}_{1_{\mathrm{x}}} = \frac{m_1}{M}\hat{P}_x + \hat{p}_x,$$

$$\hat{x}_2 = -\frac{m_1}{M}\hat{x} + \hat{X}, \qquad \hat{p}_{1_{\mathrm{y}}} = \frac{m_1}{M}\hat{P}_y + \hat{p}_y,$$

$$\hat{y}_1 = \frac{m_2}{M}\hat{y} + \hat{Y}, \qquad \hat{p}_{2_{\mathrm{x}}} = \frac{m_2}{M}\hat{P}_x - \hat{p}_x,$$

$$\hat{y}_2 = -\frac{m_1}{M}\hat{y} + \hat{Y}, \qquad \hat{p}_{2_{\mathrm{y}}} = \frac{m_2}{M}\hat{P}_y - \hat{p}_y,$$

By substituting the equation above, the Hamiltonian then becomes

$$\hat{H} = \frac{1}{2m_1}\left(\frac{m_1}{M}\hat{P}_x + \hat{p}_x\right)^2 + \frac{1}{2m_1}\left(\frac{m_1}{M}\hat{P}_y + \hat{p}_y\right)^2 + \frac{1}{2}m_1\omega^2\left(\frac{m_2}{M}\hat{x} + \hat{X}\right)^2 + \frac{1}{2}m_1\omega^2\left(\frac{m_2}{M}\hat{y} + \hat{Y}\right)^2 + \frac{1}{2m_2}\left(\frac{m_2}{M}\hat{P}_x - \hat{p}_x\right)^2 + \frac{1}{2m_2}\left(\frac{m_2}{M}\hat{P}_y + \hat{p}_y\right)^2 + \frac{1}{2}m_2\omega^2\left(-\frac{m_1}{M}\hat{x} + \hat{X}\right)^2 + \frac{1}{2}m_2\omega^2\left(-\frac{m_1}{M}\hat{y} + \hat{Y}\right)^2 + \frac{1}{2}k\hat{x}^2 + \frac{1}{2}k\hat{y}^2$$

After some simplification, the Hamiltonian then minimize to the following equation

$$\begin{aligned}\hat{H} = & \left(\frac{1}{2M}\hat{P}_x^2 + \frac{1}{2M}\hat{P}_y^2 + \frac{1}{2}M\omega^2\hat{X}^2 + \frac{1}{2}M\omega^2\hat{Y}^2\right) \\ & + \left(\frac{1}{2\mu}\hat{p}_x^2 + \frac{1}{2\mu}\hat{p}_y^2 + \frac{1}{2}\mu\Omega^2\hat{x}^2 + \frac{1}{2}\mu\Omega^2\hat{y}^2\right) \\ = & \hat{H}_{CM} + \hat{H}_{rel}\,, \end{aligned} \tag{3.2}$$

where $\hat{H}_{CM}$ is hamiltonian operator for the motion of center of mass, $\hat{H}_{rel}$ is Hamiltonian operator for the relative motion where the relative mass and frequency for relative mass is $\Omega^2 = \omega^2 + \frac{K}{\mu}$.

3.1.1 Hamiltonian for Coupled Harmonic Oscillators

The difference between the above is that 3.1.1 totally involves two masses of particle but for 3.1 it involve only one masses of particle where it focuses on the center of mass and relative mass. A system with two coupled harmonic oscillators (HOS) are being parametrized by coordinates X_1, X_2 and the masses m_1 and m_2. The Hamiltonian for the coupled HOS is then being expressed by

$$H_1 = \frac{1}{2m_1}P_1^2 + \frac{1}{2m_2}P_2^2 + \frac{1}{2}(C_1X_1^2 + C_2X_2^2 + C_3X_1X_2), \tag{3.3}$$

where C_1, C_2 and C_3 are said to be the constant parameter. The position variables are then

$$x_1 = \left(\frac{m_1}{m_2}\right)^{\frac{1}{4}} X_1, \qquad x_2 = \left(\frac{m_2}{m_1}\right)^{\frac{1}{4}} X_2,$$

and their momenta are

$$p_1 = \left(\frac{m_2}{m_1}\right)^{\frac{1}{4}} P_1, \qquad p_2 = \left(\frac{m_1}{m_2}\right)^{\frac{1}{4}} P_2.$$

After some rescalling, the mass and other constants are then redefined to be

$$m = (m_1 m_2)^{1/2}, \qquad c_1 = C_1 \sqrt{\frac{m_2}{m_1}}$$

$$c_2 = C_2 \sqrt{\frac{m_1}{m_2}}, \qquad c_3 = C_3.$$

Then, the equation of (3.3) will then become

$$H_1 = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}(c_1 x_1^2 + c_2 x_2^2 + c_3 x_1 x_2), \tag{3.4}$$

3.2 Time-Independent Schrödinger Equation of Isotropic Harmonic Oscillator

The Time-Independent Schrödinger Equation is derived from the Time-Dependent Schrödinger Equation where it separates time in that equation. Given the Schrödinger Equation is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi, \tag{3.5}$$

where the wave function of this equation is a simple product between two functions

$$\Psi(x,t) = \psi(x) f(t),$$

ψ is a function of x (point of particle) and f is a function of t (time). Then, solution with separation of variables can take place. The wave functions then becomes

$$\frac{\partial \Psi}{\partial t} = \psi \frac{df}{dt}, \frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x^2} f. \tag{3.6}$$

By inserting equation (3.6) into (3.5), the Schrödinger Equation becomes

$$i\hbar \psi \frac{df}{dt} = -\frac{\hbar^2}{2m}\frac{d^2 \psi}{dx^2} f + V\psi f, \tag{3.7}$$

then we divide equation of (3.7) with ψf, we then get

$$i\hbar \frac{1}{f}\frac{df}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V\,, \tag{3.8}$$

Through this equation we can see that the left hand side is a function of t while the other side is a function of x. The only way to make this equation to be true is both sides must be a constant. Then we assume that

$$i\hbar \frac{1}{f}\frac{df}{dt} = E, \tag{3.9}$$

where E is a separation constant. Next, we substitute the (3.9) into (3.8) to get

$$-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V = E, \tag{3.10}$$

and multiply equation (3.8) with ψ to get the Time-Independent Schrödinger Equation (Belkacemi et. al, 2000). The final form of the Time-Independent Schrödinger Equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi.$$

This part is already well known in any Quantum Mechanics textbooks.

3.2.1 Isotropic Harmonic Oscillator in a Cartesian Plane

This part is to discuss how the solution for isotropic harmonic oscillator in a cartesian plane where x and y axis are involved. Two-dimensional isotropic harmonic oscillator have been discussed in equation (3.0) where the Hamiltonian is

$$\hat{H} = \frac{1}{2m}\left(\hat{P}_{\mathrm{x}}^2 + \hat{P}_{\mathrm{y}}^2\right) + \frac{1}{2}m\omega^2(\hat{x}_1^2 + \hat{y}_1^2).$$

So, Time-Independent Schrödinger Equation will then be

$$-\frac{ħ^2}{2m}\left(\frac{\partial^2\psi(x,y)}{dx^2}+\frac{\partial^2\psi(x,y)}{dy^2}\right)+\frac{1}{2}m\omega^2(x^2+y^2)\Psi(x,y)=E\Psi(x,y), \tag{3.11}$$

where we use a method of separation of variables as the equation (3.11) are separable. Let

$$\Psi(x,y)= X(x)Y(y);$$

$$\frac{\partial^2\psi(x,y)}{dx^2}=\frac{d^2}{dx^2}(X(x)Y(y)),$$

$$\frac{\partial^2\psi(x,y)}{\partial x^2}=(Y(y))(\frac{d^2}{dx^2}X(x))+X(x)(\frac{d^2}{dx^2}Y(y)),$$

$$\frac{\partial^2\psi(x,y)}{dx^2}=Y(y)\left(\frac{d^2}{dx^2}X(x)\right). \tag{3.12}$$

The same goes to $\frac{d^2\psi(x,y)}{dx^2}$, where

$$\frac{\partial^2\psi(x,y)}{dy^2}=\frac{d^2}{dy^2}(X(x)Y(y)),$$

$$\frac{\partial^2\psi(x,y)}{dy^2}=X(x)\left(\frac{d^2}{dy^2}Y(y)\right) \tag{3.13}$$

Substitute equation (3.12) and (3.13) into equation (3.8)

$$-\frac{ħ^2}{2m}\left(Y(y)\left(\frac{d^2}{dx^2}X(x)\right)+X(x)\left(\frac{d^2}{dy^2}Y(y)\right)\right)+\frac{1}{2}m\omega^2(x^2+y^2)X(x)Y(y)$$
$$=EX(x)Y(y) \tag{3.14}$$

We then group the equation (3.14) into *x*- and *y*-dependent terms. There will be two ordinary differential equations that can be expressed based on the equation of (3.14)

$$-\frac{h^2}{2m}\frac{d^2X(x)}{dx^2}+\frac{1}{2}m\omega^2x^2X(x)=E_xX(x) \tag{3.15}$$

$$-\frac{ħ^2}{2m}\frac{d^2Y(y)}{dy^2}+\frac{1}{2}m\omega^2y^2Y(y)=E_yY(y) \tag{3.16}$$

3.2.2 Isotropic Harmonic Oscillator in a Polar Plane

In this particular part, we discuss on the same Isotropic Harmonic Oscillator but with different representation of the plane namely using the polar coordinates. Harmonic Oscillator in polar coordinates are given in (r,θ) and the Schrödinger Equation then becomes

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi(r,\theta)}{\partial\theta^2}\right)+\frac{1}{2}m\omega^2r^2\Psi(r,\theta)=E\Psi(r,\theta) \qquad (3.17)$$

Multiply the equation (3.17) with $-\frac{2m}{\hbar^2}$,

$$\frac{\partial^2\Psi(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi(r,\theta)}{\partial\theta^2}+\frac{2m}{\hbar^2}\left(E-\frac{1}{2}m\omega^2r^2\right)\Psi(r,\theta)=0, \qquad (3.18)$$

and then multiply again equation (3.16) with r^2 to disappear all the r at the denominator

$$r^2\frac{\partial^2\Psi(r,\theta)}{\partial r^2}+r\frac{\partial\Psi(r,\theta)}{\partial r}+\frac{\partial^2\Psi(r,\theta)}{\partial\theta^2}+\frac{2m}{\hbar^2}\left(E-\frac{1}{2}m\omega^2r^2\right)r^2\Psi(r,\theta)=0, \qquad (3.19)$$

as we do for equation (3.11), we will then used the separation method as the equation that we get in (3.19) is separable. Let

$$R(r)\Theta(\theta)$$

and we have

$$\frac{\partial^2\Psi(r,\theta)}{\partial r^2}=\Theta(\theta)\frac{d^2R(r)}{dr^2}+R(r)\frac{d^2\Theta(\theta)}{dr^2}=\Theta(\theta)\frac{d^2R(r)}{dr^2}; \qquad (3.20)$$

$$\frac{\partial^2\Psi(r,\theta)}{\partial r}=\Theta(\theta)\frac{dR(r)}{dr}+R(r)\frac{d\Theta(\theta)}{dr}=\Theta(\theta)\frac{dR(r)}{dr}; \qquad (3.21)$$

$$\frac{\partial^2\Psi(r,\theta)}{\partial\theta^2}=\Theta(\theta)\frac{d^2R(r)}{d\theta^2}+R(r)\frac{d^2\Theta(\theta)}{d\theta^2}=R(r)\frac{d^2\Theta(\theta)}{d\theta^2}, \qquad (3.22)$$

With the separation of variables, the equation then becomes

$$r^2\Theta(\theta)\frac{d^2R(r)}{dr^2}+r\Theta(\theta)\frac{dR(r)}{dr}+R(r)\frac{d^2\Theta(\theta)}{d\theta^2}+\frac{2m}{\hbar^2}\left(E-\frac{1}{2}m\omega^2r^2\right)r^2R\Theta=0. \qquad (3.23)$$

Divide equation (3.23) with $R(r)\Theta(\theta)$ and then we will get

$$r^2 \frac{1}{R(r)} \frac{d^2R(r)}{dr^2} + r\frac{1}{R(r)} \frac{dR(r)}{dr} + \frac{1}{\Theta(\theta)} \frac{d^2\Theta(\theta)}{d\theta^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2 r^2\right) r^2 = 0 \quad (3.24)$$

same as we did in equation (3.14). Equation (3.24) will then be grouped into r-dependent and also θ-dependent terms:

$$r^2 \frac{1}{R(r)} \frac{d^2R(r)}{dr^2} + r\frac{1}{R(r)} \frac{dR(r)}{dr} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2 r^2\right) r^2 = -\frac{1}{\Theta(\theta)} \frac{d^2\Theta(\theta)}{d\theta^2}. \quad (3.25)$$

This is the Time-Independent Schrödinger Equation where it has been grouped with two dependent of polar coordinate (r and θ).

The Cartesian coordinates (*x*,*y*) specify the location of a point in the plane in two dimensions. Polar coordinates are another two-dimensional coordinate system. Polar coordinates specify the location of a point in the plane by its distance r from the origin and the angle formed by the line segment from the origin to a point and the positive *x*-axis, rather than signed distances along the two coordinate axes.The range of r is basically from 0 to infinity while the range for the θ would be from 0 to 2π. These polar coordinates are often used for position and navigation as the direction of the travel can be given as an angle and distance from the object being considered.

3.3 Stationary Solutions of Schrödinger equations

3.3.1 In a Cartesian Plane

The energy eigenvalues based in equation of (3.15) and (3.16) are

$$E_{n_x} = \left(n_x + \frac{1}{2}\right)\hbar\omega, \quad (3.26)$$

$$E_{n_y} = \left(n_y + \frac{1}{2}\right)\hbar\omega, \quad (3.27)$$

while the eigenstate of the equation that we get using the Hermite Polynomial is then to be

$$X_{n_x}(x) = \frac{1}{\sqrt{2^{n_x} n_x!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_{n_x}\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{1}{2}\omega t},$$

$$Y_{n_y}(y) = \frac{1}{\sqrt{2^{n_y} n_y!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_{n_y}\left(\sqrt{\frac{m\omega}{\hbar}} y\right) e^{-\frac{1}{2\hbar} m\omega y^2},$$

Thus the energy levels that we can conclude according to the equation is said to be

$$E_{n_x,n_y} = E_{n_x} + E_{n_y};$$

$$E_{n_x,n_y} = \left(n_x + \tfrac{1}{2}\right)\hbar\omega + \left(n_y + \tfrac{1}{2}\right)\hbar\omega;$$

$$E_{n_x,n_y} = \left(n_x + n_y + 1\right)h\omega.$$

Then the general solution for both *x*-dependent and *y*-dependent component is

$$\Psi_{n_x,n_y}(x,y) = \frac{1}{\sqrt{2^{n_x+n_y} n_x!\, n_y!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} H_{n_x}\left(\sqrt{\frac{m\omega}{\hbar}} x\right) H_{n_y}\left(\sqrt{\frac{m\omega}{\hbar}} y\right) e^{-\frac{1}{2\hbar} m\omega(x^2+y^2)},$$

where $n_x = n_y = 0,1,2,\ldots$ are the quantum number, associated to *x* and *y* and both H_{n_x} and H_{n_y} are Hermite polynomial of the quantum number associated to *x* and *y*.

In coupled harmonic oscillator system, the energy eigenvalue and energy eigenstate would be the summation of energy for center of mass motion and energy for the relative motion

$$E = E_{CM} + E_{rel};$$

$$E = \left(n_x + n_y + 1\right)\hbar\omega + (n_X + n_Y + 1)\hbar\Omega,$$

where the component of *x* and *y* are referred to the relative mass of motion and $n_X = n_Y$ =0,1,2,... are the quantum number for relative mass of motion and the frequency of relative motion would be $\Omega^2 = \omega^2 + \frac{K}{\mu}$.

The general equation for the coupled isotropic harmonic oscillator would be

$$\Psi_{n_X,n_Y;n_x,n_y}(X,Y;x,y) = \Psi_{n_X,n_Y}(X,Y)\Psi_{n_x,n_y}(x,y)$$

$$\Psi_{n_X,n_Y;n_x,n_y}(X,Y;x,y) = \frac{1}{\sqrt{2^{n_X+n_Y+n_x+n_y} n_X!\, n_Y!\, n_x!\, n_y!}} \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{\mu\Omega}{\pi\hbar}\right)^{\frac{1}{2}}$$

$$H_{n_X}\left(\sqrt{\frac{M\omega}{\hbar}}X\right) H_{n_Y}\left(\sqrt{\frac{M\omega}{\hbar}}Y\right) H_{n_x}\left(\sqrt{\frac{\mu\Omega}{\hbar}}x\right) H_{n_y}\left(\sqrt{\frac{\mu\Omega}{\hbar}}y\right)$$

$$e^{-\frac{1}{2\hbar}M\omega\left(X^2+Y^2\right)-\frac{1}{2h}\mu\Omega\left(x^2+y^2\right)}$$

3.3.2 In a Polar Plane

By solving the equation (3.25) with separation of variable method (SVM), both of the dependent must matched to a fixed value. We let m_l^2 as a constant

$$r^2 \frac{1}{R(r)} \frac{d^2R(r)}{dr^2} + r \frac{1}{R(r)} \frac{dR(r)}{dr} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2} m\omega^2 r^2\right) r^2 = m_l^2 \tag{3.28}$$

$$-\frac{1}{\Theta(\theta)} \frac{d^2\Theta(\theta)}{d\theta^2} = m_l^2 \tag{3.29}$$

Based on the equation (3.24),

$$\frac{d^2\Theta(\theta)}{d\theta^2} + m_l^2 \Theta(\theta) = 0,$$

we let $\frac{d^2}{d\theta^2} = \mathrm{D}^2$ to find the roots

$$\mathrm{D}^2 \Theta(\theta) + m_l^2 \Theta(\theta) = 0;$$

$$(\mathrm{D}^2 + m_l^2)\Theta(\theta) = 0;$$

$$\mathrm{D}^2 + m_l^2 = 0;$$

$$\mathrm{D} = +im_l\, , -im_l.$$

To solve this we need to use Ordinary Differential Equation (ODE)

$$\Theta(\theta) = A\cos\, m_l\theta + B\sin\, m_l\theta\, ,$$

equivalently

$$\Theta(\theta) = Ae^{-im_l\theta} + Be^{im_l\theta}\, .$$

To make $\Theta(\theta)$ must be single valued, then they have to satisfy the equation

$$e^{im_l\theta} = e^{im_l(\theta+2\pi)}$$

(Pipes & Boas, 1967). Then let $\theta = 0$,

$$1 = e^{i2\pi m_l}.$$

Then the value of angular momentum, $m_l = 0, \pm 1, \pm 2, \ldots$. After that we solved for r-dependence . The equation of (3.28) is then divided with r^2 and multiplied with $R(r)$

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2 r^2\right)R(r) - \frac{m_l^2}{r^2}R(r) = 0,$$

For the solution of above equation , we got the eigenstate is

$$R(r) = N_{n_r,m_l} r^{|m|} e^{-\frac{r^2}{2\lambda^2}} L_{n_r}^{|m_l|}\left(\frac{r^2}{\lambda^2}\right).$$

The energy eigenvalue for the center mass of motion in isotropic harmonic oscillator is

$$E_{n_r,m_l} = (2n_r + |m_l| + 1)\hbar\omega,$$

where radial, $n_r = 0,1,2, \ldots$ is quantum number related with the center of mass motion and the angular momentum, $m_l = 0, \pm 1, \pm 2, \ldots$. So, the general solution for this polar plane in terms of radial and angular momentum is

$$\Psi_{n_r,m_l}(r,\theta) = R_{m_l}(r)\Theta_{n_r}(\theta),$$

$$\Psi_{n_r,m_l}(r,\theta) = N_{n_r,m_l} r^{|m|} e^{-\frac{r^2}{2\lambda^2}} L_{n_r}^{|m_l|}\left(\frac{r^2}{\lambda^2}\right) e^{im_l\theta}.$$

with

$$N_{n_r,m_l} = \frac{1}{\lambda^{|m_l|+1}}\sqrt{\frac{n_r!}{\pi(n_r + |m_l|)!}}$$

where N_{n_r,m_l} is a normalization constant and $L_{n_r}^{|m_l|}$ are solved using the associated Laguerre polynomials. This associated Laguerre polynomials is defined as

$$L_n^k(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x),$$

and the $L_{n+k}(x)$ can be solved with the Laguerre polynomials where it was defined by Rodrigues Formula (Moss, 1992).

We then further this study with coupled isotropic harmonic oscillator where the energy eigenvalue for the center of mass motion and relative motion would be

$$E = E_{CM} + E_{\mathrm{rel}}\,;$$

$$E = (2n_r + |m_l| + 1)\hbar\omega + (2n_R + |m_L| + 1)\hbar\Omega\,,$$

where $n_R = 0,1,2,\ldots$ is radial quantum numbers associated with the relative motion and angular momentum quantum number is given as $m_L = 0, \pm 1, \pm 2, \ldots$.

The general equation for the coupled harmonic oscillator would be

$$\Psi_{n_R,m_L;n_r,m_l}(r_{CM},\theta_{CM};r_{rel},\theta_{rel}) = \Psi_{n_R,m_L}(r_{CM},\theta_{CM})\Psi_{n_r,m_l}(r_{rel},\theta_{rel})$$

$$\Psi_{n_R,m_L;n_r,m_l}(r_{CM},\theta_{CM};r_{rel},\theta_{rel}) = N_{n_R,m_L}N_{n_r,m_l}r_{CM}^{|m_L|}r_{rel}^{|m_l|}e^{-\frac{r_{CM}^2}{2\lambda_1^2}}e^{-\frac{r_{rel}^2}{2\lambda_2^2}}$$

$$L_{n_R}^{|m_L|}\left(\frac{r_{CM}^2}{\lambda_1^2}\right)L_{n_r}^{|m_l|}\left(\frac{r_{rel}^2}{\lambda_2^2}\right)e^{im_L\theta_{CM}}e^{im_l\theta}$$

where N_{n_R,m_L} is also a normalization constant and $L_{n_R}^{|m_L|}$ is the generalized Laguerre's polynomials and $\lambda_1 = \sqrt{\frac{\hbar}{M\omega}}$ with $\lambda_2 = \sqrt{\frac{\hbar}{\mu\Omega}}$ are the natural scales for length.

3.4 Noncommutative Phase Space

As the study is the visualisation of the coupled harmonic oscillator in noncommutative phase space, we have to discussed more on the noncommutative geometry. This noncommutative geometry is used in two coupled harmonic oscillator where we impose the coordinates of plane do not commute

$$[x_i, x_j] = i\theta_{ij}, \tag{3.30}$$

with $\theta_{ij} = \epsilon_{ij}\theta$ is noncommutativity parameter. To obtain this relation, star-product definition is used

$$f(x) \star g(x) = \exp\left\{\frac{i}{2}\theta_{ij}\,\partial_x\,\partial_y\right\} f(x)g(y)\Big|_{x=y} \tag{3.31}$$

where f and g are two arbitrary functions. Then, we also use the standard commutation relations which is

$$[p_i, x_j] = -i\hbar\delta_{ij}, \qquad [p_i, p_j] = 0. \tag{3.32}$$

3.5 Creation and Annihilation Operators

Creation and annihilation operators, also known as ladder operators, whose actions are the addition and subtraction of fixed quanta of energy to the oscillator system in quantum physics. The ladder operator is denoted by $a_{\pm}$, where a_+ is referred to as the raising operator and the a_- is referred to as the lowering operator. It enables eigenvalues to go up and down the energy scale. The time-independent Schrödinger equations, (3.10) as the potential energy, $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$ can be defined in terms of the creation and annihilation operators which are

$$\left(a_-a_+ - \frac{1}{2}\hbar\omega\right)\psi = E\psi. \tag{3.33}$$

and

$$\left(a_+a_- - \frac{1}{2}\hbar\omega\right)\psi = E\psi \tag{3.34}$$

where $a_{\pm}$ is given as below

$$a_{\pm} \equiv \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} \pm im\omega x\right). \tag{3.34}$$

From equations (3.33) and (3.34), we can see that the order of the factors a_+ and a_- is very important as they can change the sign of the The time-independent Schrödinger equations.

3.6 Visualizing Eigenstate of The Groundstate Wavefuntion of Coupled Harmonic Ocillator in Noncommutative Phase Space

This simulation on the visualisation of the wavefunction of the Coupled Harmonic Oscillator in Noncommutative Phase space is generated by Mathematica 12.0 and were used to verify and to plot the Coupled Harmonic Oscillator on Noncommutative Plane. Three-Dimensional Plotting (3-D Plotting) helps us in visualising the particle's wavefunction with respect to three axes in the attempt to show the relationchip between three variables. Other than that, we also used density plot where it displays three dimensional data in two dimensions using contour and color-coded regions.

CHAPTER 4

SOLUTIONS OF COUPLED HARMONIC OSCILLATOR IN NONCOMMUTATIVE PHASE SPACE

In chapter 3, we have learned and calculated the hamiltonian for the harmonic oscillator and coupled harmonic oscillator. In this chapter, we have taken into account the Noncommutative phase space and derive the Hamiltonian so that the equations are suitably based on the Noncommutative Phase Space.

4.1 Coupled Harmonic Oscillator in Noncommutative Phase Space

Furthering our study, we used noncommutative geometry as we impose the coordinates of the plane do not commute. First of all, defining the Hamiltonian (3.4) on $\mathbb{R}^2_\theta$ and the H_1 acts on arbitrary functions $\Psi(\mathbf{r}, t)$ as below

$$H_1 \star \Psi(\mathbf{r}, t) = H_1^{\text{nc}} \Psi(\mathbf{r}, t), \tag{4.1}$$

where we use the definition of the star-product in (3.31) to obtain the Hamiltonian of the coupled harmonic oscillator in noncommutative phase space. The relationship between the coordinate of x_1 and x_2 and the momentum, p_1 and p_2 is formulated as below

$$x_1 = x_1 - \frac{\theta}{2\hbar} p_2, \qquad x_2 = x_2 + \frac{\theta}{2\hbar} p_1, \tag{4.2}$$

and subtitute equation (4.2) into the Hamiltonian (3.4) where we obtain

$$\begin{aligned} H_1^{\text{nc}} = \; & \frac{1}{2m}(p_1^2 + p_2^2) + \frac{c_1}{2}\left(x_1 - \frac{\theta}{2\hbar} p_2\right)^2 \\ & + \frac{c_2}{2}\left(x_2 + \frac{\theta}{2\hbar} p_1\right)^2 + \frac{c_3}{2}\left(x_1 - \frac{\theta}{2\hbar} p_2\right)\left(x_2 + \frac{\theta}{2\hbar} p_1\right). \end{aligned} \tag{4.3}$$

4.2 Transformation of a new phase space variables

With the involvement of the interacting term, the new phase space variables transformations must take place where

$$y_a = M_{ab} x_b, \qquad \hat{p}_a = M_{ab} p_b, \tag{4.4}$$

where the matrix, M_{ab} is a unitary rotation are given as below

$$(M_{ab}) = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \tag{4.5}$$

with angle, α. Equation (4.3) will then be inserted into equation (4.2) where α must satisfy the condition of

$$\tan\,\alpha = \frac{c_3}{c_2 - c_1} \tag{4.6}$$

and the Hamiltonian can be factorized into this form as in equation (4.6)

$$H_2^{\mathrm{nc}} = e^{\eta}\mathcal{H}_1 + e^{-\eta}\mathcal{H}_2, \tag{4.7}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ is two commuting parts which are

$$\mathcal{H}_1 = \frac{1}{2M}e^{-\eta}\hat{p}_1^2 + \frac{K}{2}e^{\eta}y_1^2 + \frac{K\theta}{2\hbar}(e^{-\eta}y_2\hat{p}_1),$$
$$\mathcal{H}_2 = \frac{1}{2M}e^{\eta}\hat{p}_2^2 + \frac{K}{2}e^{-\eta}y_2^2 - \frac{K\theta}{2\hbar}(e^{\eta}y_1\hat{p}_2), \tag{4.8}$$

The Hamiltonian $\mathcal{H}^{\mathrm{nc}}$ can be derived by using the condition of

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2;$$

to become

$$\mathcal{H}^{\mathrm{nc}} = \frac{1}{2M}(e^{-\eta}\hat{p}_1^2 + e^{\eta}\hat{p}_2^2) + \frac{K}{2}(e^{\eta}y_1^2 + e^{-\eta}y_2^2) + \frac{K\theta}{2\hbar}(e^{-\eta}y_2\hat{p}_1 - e^{\eta}y_1\hat{p}_2), \tag{4.9}$$

where it is given that

$$K = \sqrt{c_1 c_2 - c_3^2/4}, \qquad e^{\eta} = \frac{c_1 + c_2 + \sqrt{(c_1 - c_2)^2 + c_3^2}}{2K},$$

and they must fulfil the condition of $4c_1c_2 > c_3^2$ because we did not want the value of K^2 to be negative. The effective mass would be

$$M = \frac{m}{1 + \left(\frac{m\omega\theta}{2\hbar}\right)^2}.$$

Furthermore, the NC system exhibits an effective mass M that is identical to the mass m for $\theta = 0$.Thus, equation (4.7) can be arranged into this form

$$\mathcal{H}'^{\text{nc}} = \frac{1}{2M}\left(\hat{P}_1^2 + \hat{P}_2^2\right) + \frac{K}{2}\left(Y_1^2 + Y_2^2\right) + \frac{K\theta}{2\hbar}\left(Y_2\hat{P}_1 - Y_1\hat{P}_2\right),$$

but then we have to rescale new coordinate of Y_i and $\hat{P}_i$, where

$$\mathcal{H}'^{\text{nc}} = \frac{1}{2M}\left(\hat{P}_1^2 + \hat{P}_2^2\right) + \frac{K}{2}\left(Y_1^2 + Y_2^2\right) + \frac{K\theta}{2\hbar}\left(Y_2\hat{P}_1 - Y_1\hat{P}_2\right),$$

and when we compare $\mathcal{H}'^{\text{nc}}$ to $\mathcal{H}$, we can see that $\mathcal{H}'^{\text{nc}}$ contains an additional term which is proportional to the θ which is basically the angular momentum.

4.3 Creation and Annihilation Operators

The expressions of position and momentum variables have to be in terms of creation and annihilation operators for the diagonalization of $\mathcal{H}'^{\text{nc}}$

$$Y_i = \sqrt{\frac{\hbar\Omega}{2K}}\left(b_i + b_i^{\dagger}\right), \qquad Q_i = i\sqrt{\frac{M\hbar\Omega}{2}}\left(b_i^{\dagger} - b_i\right), \tag{4.10}$$

and another set of operators also has been introduced to help with the mathematical calculation which is

$$B_1 = \frac{1}{\sqrt{2}}(b_1 + ib_2), \quad B_1^\dagger = \frac{1}{\sqrt{2}}\left(b_1^\dagger - ib_2^\dagger\right),$$
$$B_2 = \frac{1}{\sqrt{2}}(-b_1 + ib_2), \quad B_2^\dagger = \frac{1}{\sqrt{2}}\left(-b_1^\dagger - ib_2^\dagger\right), \tag{4.11}$$

where the effective frequency, Ω is

$$\Omega = \sqrt{\frac{K}{M}}\ .$$

The expressions of (4.7) and (4.8) must satisfy the commutation relation (see Appendix A.2 for proof)

$$\left[b_i, b_j^\dagger\right] = \delta_{ij}\ , \qquad \left[B_i, B_j^\dagger\right] = \delta_{ij}\ . \tag{4.12}$$

The equation for Hamiltonian in Noncommutative Phase Space, $\mathcal{H}'^{\mathrm{nc}}$ can then be modified into this form

$$\mathcal{H}'^{\mathrm{nc}} = \hbar\Omega_1 B_1^\dagger B_1 + \hbar\Omega_2 B_2^\dagger B_2 + \hbar\Omega, \tag{4.13}$$

where the frequency of Ω_1 and Ω_2 are given as

$$\Omega_1 = \Omega + \frac{K\theta}{2\hbar}, \qquad \Omega_2 = \Omega - \frac{K\theta}{2\hbar}. \tag{4.14}$$

4.4 Eigenequations and Eigenvalues

Equation (4.9) can be used to solve to find the eigenequations and eigenvalues where

$$\mathcal{H}'^{\mathrm{nc}}|n_1, n_2, \theta\rangle = \mathcal{E}'^{\mathrm{nc}}_{n_1,n_2}|n_1, n_2, \theta\rangle, \tag{4.15}$$

and the eigenstates would be

$$|n_1, n_2, \theta\rangle = \frac{\left(B_1^\dagger\right)^{n_1}\left(B_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0{,}0, \theta\rangle,$$

while the eigenvalues would be

$$\mathcal{E}'^{\mathrm{nc}}_{n_1,n_2} = \hbar\Omega_1 n_1 + \hbar\Omega_2 n_2 + \hbar\Omega\,,$$

$$\mathcal{E}'^{\mathrm{nc}}_{n_1,n_2} = \hbar\left(\Omega + \frac{K\theta}{2\hbar}\right)n_1 + \hbar\left(\Omega - \frac{K\theta}{2\hbar}\right)n_2 + \hbar\left(\sqrt{\frac{K}{M}}\right),$$

where the values of frequencies of Ω_1 and Ω_2 are given by (4.14). Equation (4.15) is then

$$\mathcal{H}'^{\mathrm{nc}}\frac{\left(B_1^\dagger\right)^{n_1}\left(B_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0{,}0,\theta\rangle = \hbar\left(\Omega + \frac{K\theta}{2\hbar}\right)n_1 + \hbar\left(\Omega - \frac{K\theta}{2\hbar}\right)n_2 + \hbar\left(\sqrt{\frac{K}{M}}\right)\frac{\left(B_1^\dagger\right)^{n_1}\left(B_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0{,}0,\theta\rangle\,, \tag{4.16}$$

The general wavefunction in the *x*-representation is given as follows

$$\psi_0(\mathbf{x},\theta) = \sqrt{\frac{M\Omega}{\pi\hbar}}\exp\left\{-\frac{M\Omega}{2\hbar}\left[e^{\eta}\left(x_1\cos\frac{\alpha}{2} - x_2\sin\frac{\alpha}{2}\right)^2 + e^{-\eta}\left(x_1\sin\frac{\alpha}{2} + x_2\cos\frac{\alpha}{2}\right)^2\right]\right\}, \tag{4.17}$$

which is projecting $|0{,}0,\theta\rangle$ on the plane (Y_1, Y_2) we find the ground state wave function

$$\psi_0(\mathbf{Y},\theta) = \sqrt{\frac{M\Omega}{\pi\hbar}} \exp\left\{-\frac{M\Omega}{2\hbar}(Y_1^2 + Y_2^2)\right\}.$$

The generalized wavefunction in the x-representation , (4.17) is used to describe the physical quantities of the system.

CHAPTER 5

RESULTS AND DISCUSSION

Simulating the general wavefunction of coupled harmonic oscillator in noncommutative phase space in Mathematica which it can help us to visualize the eigenequations and eigenvalues of the wavefunctions. First of all, we have to solve the eigenvalues, the eigenequations and the energy spectrum where it depend on noncommutativity parameter θ. In this study, we have to note that the mass that were used are mass of proton which is 1.673×10^{-27} kg and the mass of electron, 9.109×10^{-31}kg.

To make the visualisation becomes simpler, the noncommutativity parameter θ must be equal to zero as we assume that is the standard case for noncommutative phase space. Also, we can determine the ground state wave function of noncommutative system before it undergoes transformation .This ground state wave function was assume as the initiation point of this study.

5.1 Three Dimensional Visualisation of Coupled Harmonic Oscillator in Noncommutative Phase Space

Below are the contour plot diagrams of the wavefunction of the coupled harmonic oscillator system in noncommutative phase space. In Figure 5.1, the contour plot of the coupled harmonic oscillator in noncommutative phase space shows the different colors. Each color represents the slope of the graph. As you can see, the distance or the width for the blue region is large which means that the slope around that area is very slight so this area in 3D plot is a flat region. As we moved near to the centre, the colors keep changing as the slope becomes steeper and steeper. This can be verified in 3D plotting in Figure 5.2

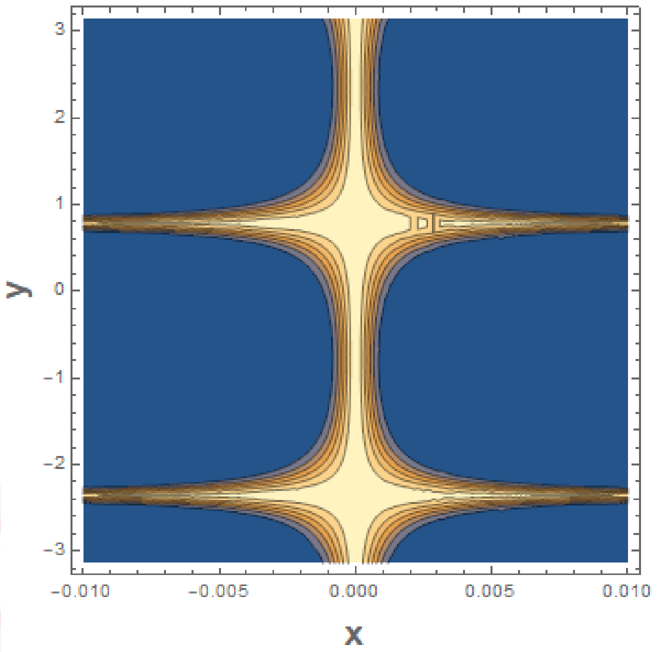


Figure 5.1 : Contour Plot of Coupled Harmonic Oscillator's Wavefunction in Noncommutative Phase Space

For plotting, let us consider the range of constant parameters, C_1, C_2 and C_3 would be in between 1×10^{-29} and 1×10^{-30}. So we will arbitrarily consider all the constant parameter to be the same value which is 1×10^{-30} . This is because we do not want the the frequency, Ω in noncommutative phase space would be much more higher. In actuality, Ω has a large influence on the generalised wavefunction because of the unmitigated massiveness of the exponent term (4.13), which utterly overwhelms the other terms preceding it. It also means that higher value of Ω would bring the negative value to exponent terms and the simulation cannot be done clearly.

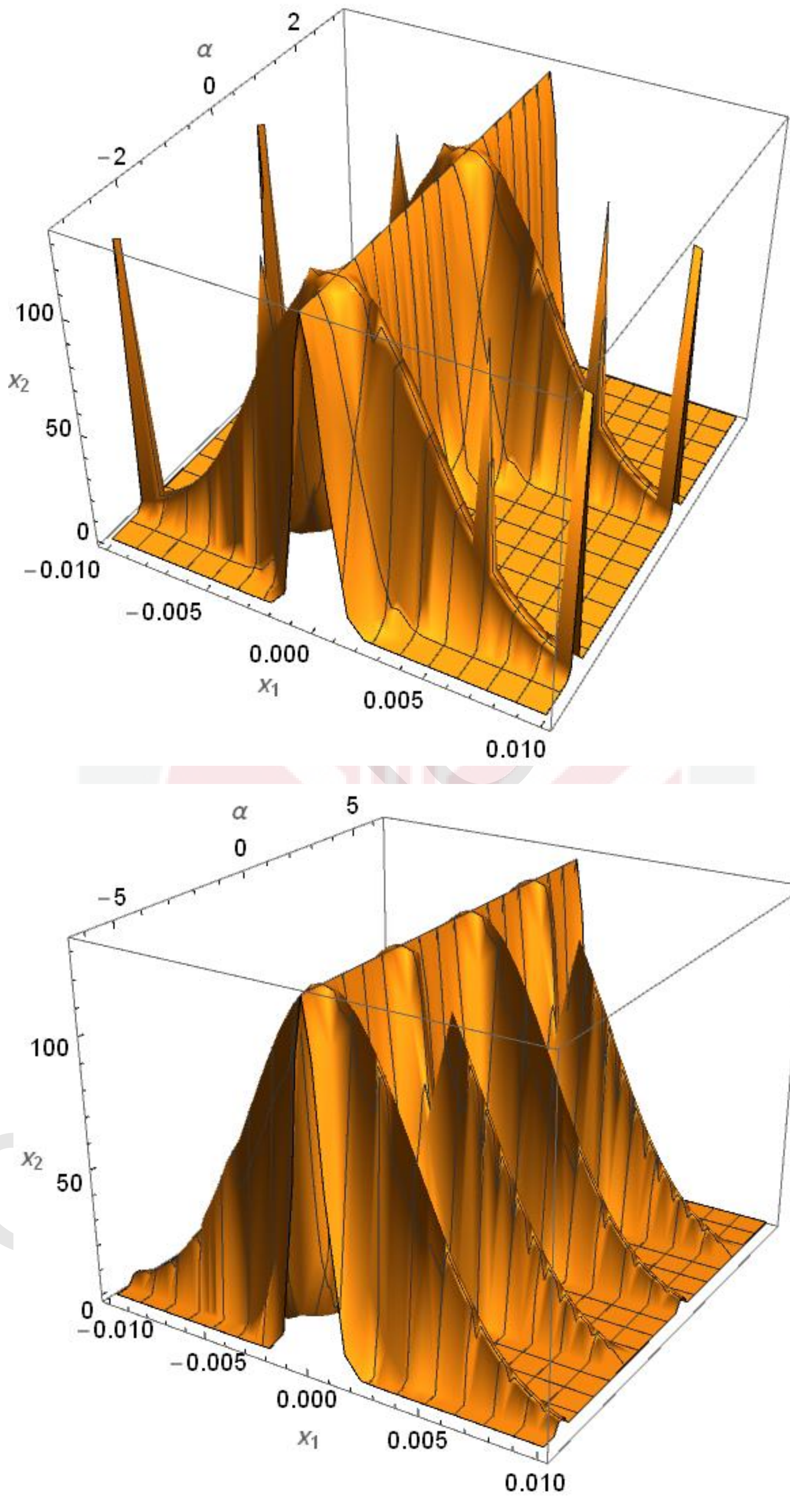


Figure 5.2: Three Dimensional (3-D) Plotting of Coupled Harmonic Oscillator's Wavefunction in Noncommutative Phase Space

By using Mathematica, the 3-D plotting of the generalized wavefunction of coupled harmonic oscillator at the ground level $|0,0,\theta\rangle$ is drawn in Figure 5.2. Based on the observation, the maximum peaks on the graph would be at the coordinate $x_1 = 0$ as shown clearly in Figure 5.3.

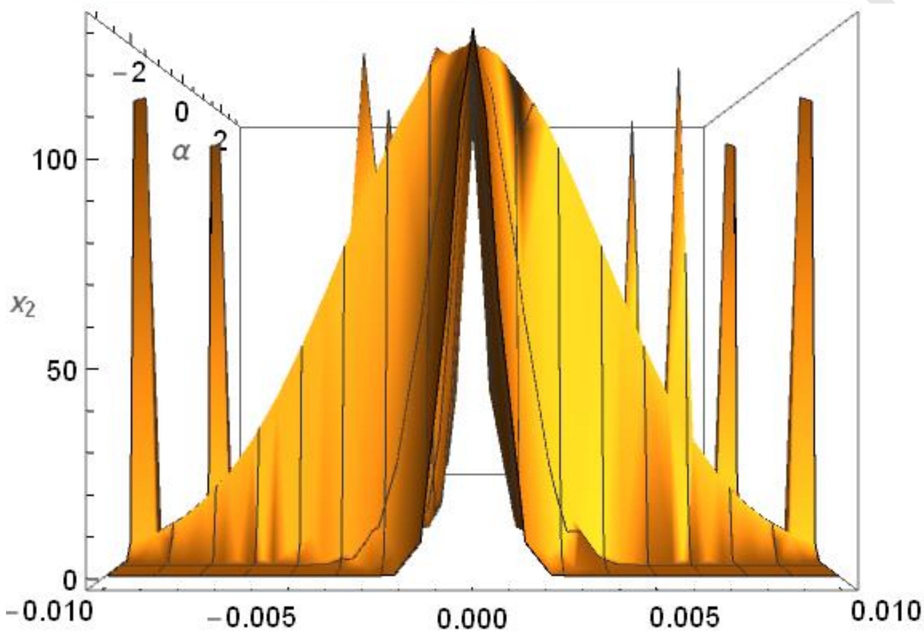
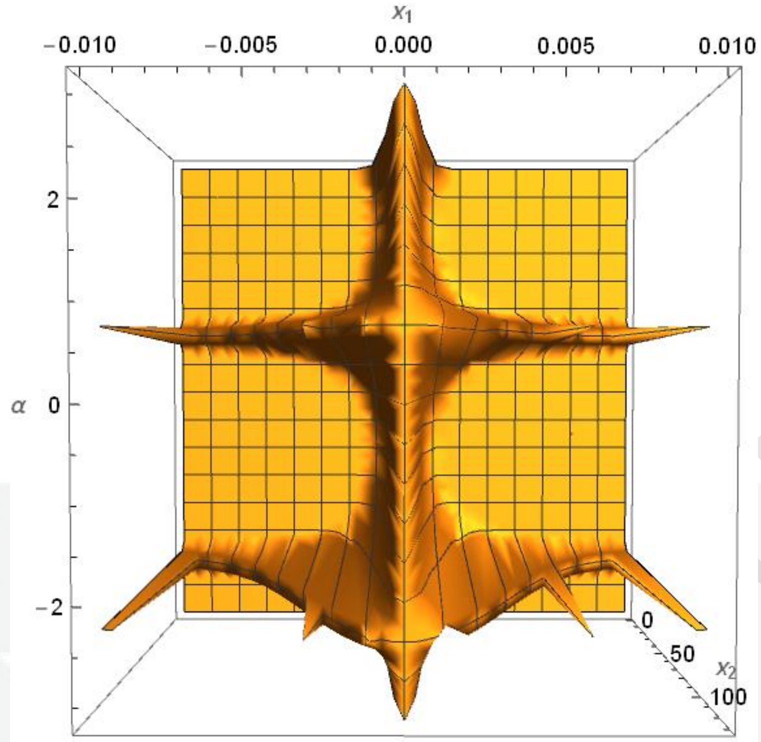
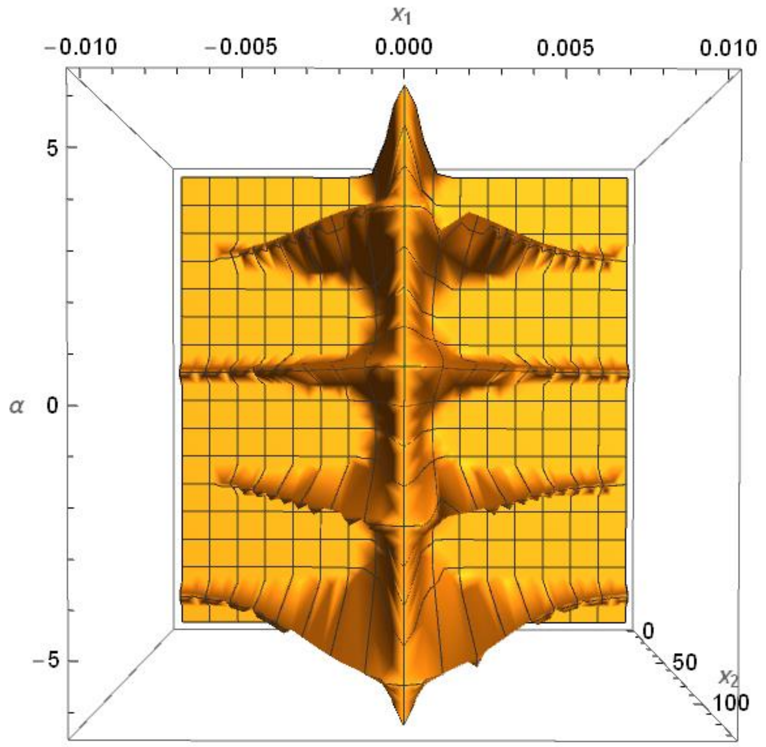


Figure 5.3: The Front View of Three Dimensional (3-D) Plotting of Coupled Harmonic Oscillator's Wavefunction in Noncommutative Phase Space

Moreover the coordinate of $x = 0.010$ and -0.010 is relatable to the angle, α as it also shown another two peaks that occur which are the range of angle, α in between 0.5 to 1.0 radians and -2.0 to -2.5 radians. The increase in range of α would increase the number of peaks which could be observed in the Figure 5.4.

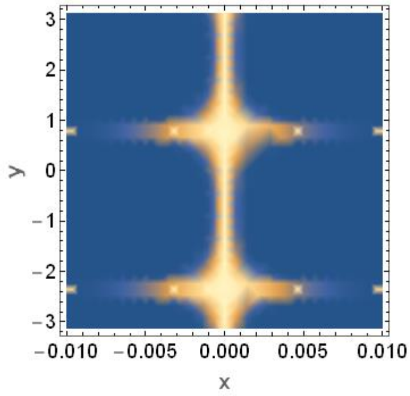


i. 3D Plotting In The Range of −π to π (radians)

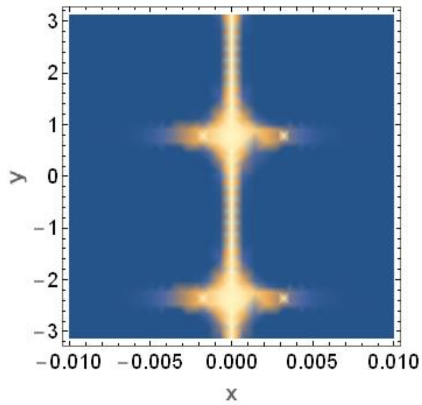


ii. 3D Plotting In The Range of −2π to 2π (radians)

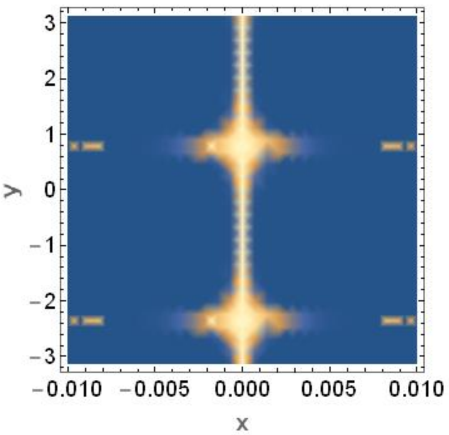
Figure 5.4: The Top View of (i) and (ii) of The Three Dimensional (3-D) Plotting of Coupled Harmonic Oscillator's Wavefunction in Noncommutative Phase Space



a) $\psi_0(\mathbf{x},\theta)$ at $\Omega = \frac{1}{4}$



b) $\psi_0(\mathbf{x},\theta)$ at $\Omega = \frac{1}{2}$



c) $\psi_0(\mathbf{x},\theta)$ at $\Omega = \frac{3}{4}$

Figure 5.5 : Density Plot For Generalized Wavefunction of Coupled Harmonic Oscillator In Noncommutative Phase Space for Several Choice of Ω

The density plot of the generalized wavefunction at the ground level is shown in Figure 5.5 for various values of Ω in the range of 0 Hz to 1 Hz only. Observe that regardless of the value of Ω, the function's maximum turning point will always be concentrated at the origin, as indicated by the brightest spot on the plot, implying that the particle will most likely be found there. The function's distribution is similarly radially symmetric as you can see in the figure 5.5, and as the frequency, Ω grows, the bright region of the plot narrows radially inward, implying that the probability of detecting the particle further away from the origin diminishes.

We assumed noncommutativity parameter, θ to be equal to 0 as we only consider this as a standard case for noncommutative phase space. So the generalized wave function for the ground state was not affected with the noncommutative parameter. The only parameter that affect this visualisation is effective frequency as has been discussed in equation (4.14). The higher the effective frequency, the lower the probabilty of detecting the particle further away from the origin.

CHAPTER 6

CONCLUSION

6.1 Conclusion

As a conclusion, to construct the generalized solution of wave function of a particle in coupled harmonic oscillator in noncommutative phase space, we need to ensure first that the spatial coordinates must not commute. The star-product definition were then applied to examine quantum mechanically a system with two linked harmonic oscillator on noncommutative phase space. The diagonalized system was then arose as the mixing angle, α of unitary rotation and Noncommutative (NC) Hamitonian was taken into account in this study. The eigenstates and energy spectrum, as well as the ground state wave function, are shown to be dependent on the noncommutativity parameter and to coincide for $\theta = 0$ with those for the standard case. This was used to specifically to define the NC system's ground state wave function prior to transformation, which was the beginning point of our investigation.

Following that, we used the tools mentioned above to plot the three-dimensional wavefunction of the coupled harmonic oscillator. Thus, we studied the physical quantities relating to the ground state by using the density plot and contour plot . According to the visualisation and simulation that was done, we can concluded that more particle can be detect in the center of the plotting. Other than that, we learned that the frequency affected the probability of detecting the particle as the higher the frequency is, less particles can be detected.

6.2 Future work

One could extend the problem of generating the general wavefunction of solution for higher quantum number state and visualised it in three dimensional plots. Other than that, study about another potential models that use this physical system described by two coupled harmonic oscillators such as the Lee model in quantum field theory, the Bogoliubov transformation in superconductivity, two-mode compressed states of light, the covariant harmonic oscillator model for the parton image, and models in molecular physics can be so much interesting as we could get to see how does the noncommutative phase space reacts on all of the models. There are also other models which have not been studied enough as one of the variables of the models is not directly observable which are Thermo-field dynamics, two-mode squeezed states, the hadronic temperature, and the Barnet–Phoenix version of information theory.

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APPENDIX A
ALGORITHMS

A.1 STIMULATED ANNEALING

In[3]:= $h = 1.055 * 10^{-34}$

Out[3]= 1.055×10^{-34}

In[5]:= $m1 = 9.109 * 10^{-31}$

Out[5]= 9.109×10^{-31}

In[6]:= $m2 = 1.673 * 10^{-27}$

Out[6]= 1.673×10^{-27}

In[7]:= $C1 = 1 * 10^{-30}$

Out[7]= $\frac{1}{1\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000}$

In[8]:= $C2 = 1 * 10^{-30}$

Out[8]= $\frac{1}{1\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000}$

In[9]:= $C3 = 1 * 10^{-30}$

Out[9]= $\frac{1}{1\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000}$

In[10]:= $m = \sqrt{m1 * m2}$

Out[10]= 3.90376×10^{-29}

In[11]:= $c1 = C1 \sqrt{\frac{m2}{m1}}$

Out[11]= 4.28561×10^{-29}

In[12]:= $c2 = C2 \sqrt{\frac{m1}{m2}}$

Out[12]= 2.33339×10^{-32}

In[14]:= c3 = C3

Out[14]= $\frac{1}{1\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000}$

In[15]:= $K = \sqrt{(c1 * c2) - \frac{c3^2}{4}}$

Out[15]= 8.66025×10^{-31}

In[16]:= $\omega = \sqrt{\frac{K}{m}}$

Out[16]= 0.148944

In[17]:= $\theta = 2 * 10^{-20}$

Out[17]= $\frac{1}{50\,000\,000\,000\,000\,000\,000}$

In[18]:= $M = \frac{m}{1 + \left(\frac{m * \omega * \theta}{2\,h}\right)^2}$

Out[18]= 3.90376×10^{-29}

In[19]:= $w = \sqrt{\frac{K}{M}}$

Out[19]= 0.148944

In[20]:= $d = \frac{\left(c1 + c2 + \sqrt{(c1 - c2)^2 + c3^2}\right)}{2\,K}$ // FullSimplify

Out[20]= 49.4927

In[22]:= $f = \frac{\left(c1 + c2 - \sqrt{(c1 - c2)^2 + c3^2}\right)}{2\,K}$ // FullSimplify

Out[22]= 0.020205

$$\text{Plot3D}\left[\sqrt{\frac{(M \times w)}{\pi \times h}} \left(e^{\frac{-(M \times w)}{2\,h}\left(d\,(x\,\text{Cos}[y] - x\,\text{Sin}[y])^2 + f\,(x\,\text{Sin}[y] + x\,\text{Cos}[y])^2\right)}\right), \{x, -1 * 10^{-2}, 1 * 10^{-2}\}, \{y, -2\pi, 2\pi\},\right.$$

```
PlotRange → Full,
AxesLabel → {Style["x1", Bold, 18], Style["α", Bold, 18], Style["x2", Bold, 18]},
TicksStyle → Directive[Black, 18],
BoxRatios → {2, 2, 1.5},
ImageSize → Large]
```

In[50]:=

```
ContourPlot[Sqrt[(M× w)/(π × h)] (e^(-(M×w)/(2 h)) (d (x Cos[y] - x Sin[y])^2 + f (x Sin[y] + x Cos[y])^2)), {x, -1 * 10^-2, 1 * 10^-2}, {y, -π, π},
 FrameLabel -> {Style["x", Bold, 18], Style["y", Bold, 18]}]

Manipulate[DensityPlot[(Sqrt[(M× w)/(π × h)] e^(-(M×w)/(2 h)) (d (x Cos[y] - x Sin[y])^2 + f (x Sin[y] + x Cos[y])^2)), {x, -1 * 10^-2, 1 * 10^-2}, {y, -π, π},
  FrameLabel -> {Style["x", Bold, 18], Style["y", Bold, 18]},
  FrameTicks → {{{-1 * 10^-2, 0, 1 * 10^-2}, Automatic}, , {{-π, 0, π}, Automatic}},
  FrameTicksStyle → Directive[Black, 18],
  ImageSize → Medium,
  PlotRange → All, Ticks → {{-1 * 10^-2, 0, 1 * 10^-2}, {-π, 0, π}},
  PerformanceGoal → "Quality", ImageSize → Medium], {w, 1/4, 3/4, 1/4}]
```

A.2 COMMUTATION RELATION

$$B_1 = \frac{1}{\sqrt{2}}(b_1 + ib_2), \qquad B_1^\dagger = \frac{1}{\sqrt{2}}(b_1^\dagger - ib_2^\dagger),$$
$$B_2 = \frac{1}{\sqrt{2}}(-b_1 + ib_2), \quad B_2^\dagger = \frac{1}{\sqrt{2}}(-b_1^\dagger - ib_2^\dagger), \tag{4.11}$$

$$[b_i, b_j^\dagger] = \delta_{ij}\,, \qquad [B_i, B_j^\dagger] = \delta_{ij}\,. \tag{4.12}$$

To verify this commutation relation, we must than have the commutative properties which

$$[\mathrm{A}, \mathrm{B}] = \mathrm{AB} - \mathrm{BA}\ .$$

Before we verify $[B_i, B_j^\dagger] = \delta_{ij}$, $[b_i, b_j^\dagger] = \delta_{ij}$ have to be solve as below

$$[b_i, b_i^+] = AB - BA$$
$$= (b_i b_j^+ - b_j^+ b_i).$$

So since $\mathrm{i} = \mathrm{j} \to 1$, the equation above will be

$$b_i b_i^+ - b_i^+ b_i = 1$$

and

$$b_j b_j^+ - b_j^+ b_j = 1.$$

For i ≠ j, the equation will be

$$b_i b_j^{+} - b_j^{+} b_i = 0;$$

$$b_j b_i^{+} - b_i^{+} b_j = 0\ ;$$

To become

$$b_i b_j^{+} = b_j^{+} b_i;$$

$$b_j b_i^{+} = b_i^{+} b_j\ .$$

Then we solve for $\left[B_1, B_1^{\dagger}\right]$ as shown below

$$[B_1, B_1^{+}] = AB - BA$$

$$= \left(\left(\frac{1}{\sqrt{2}}(b_1 + ib_2)\right)\left(\frac{1}{\sqrt{2}}(b_1^{\dagger} - ib_2^{\dagger})\right)\right) - \left(\left(\frac{1}{\sqrt{2}}(b_1^{\dagger} - ib_2^{\dagger})\right)\left(\frac{1}{\sqrt{2}}(b_1 + ib_2)\right)\right)$$

$$= \left(\frac{1}{2}(b_1 b_1^{+} - ib_1 b_2^{+} + ib_2 b_1^{+}, -i^2 b_2 b_2^{+})\right) - \left(\frac{1}{2}({b_1}^{+} b_1 + i{b_1}^{+} b_2 - i{b_2}^{+} b_1 - i^{2} {b_2}^{+} b_2)\right)$$

$$= \frac{1}{2}\left(1 + {b_1}^{+} b_1 - i({b_2}^{+} b_1) + i(b_2 b_1^{+}) - (-1)(1 + {b_2}^{+} b_2)\right) - \frac{1}{2}({b_1}^{+} b_1 + i(b_2 {b_1}^{+}) - (i{b_2}^{+} b_1) - (-1)({b_2}^{+} b_2)$$

$$= 1$$

same goes to $\left[B_2, B_2^{\dagger}\right]$. For $\left[B_1, B_2^{\dagger}\right]$ and $\left[B_2, B_1^{\dagger}\right]$ the value will become zero. So it satisfied the equation in (4.12)